$$\begin{cases} u_0 = 1 \\ u_{n+1} = \dfrac{u_n^2}{2+u_n} \quad ; \quad n \in \mathbb{N} \end{cases}$$

لدينا $u_0 = 1$ إذن $0 \le u_0 \le 1$

- نفترض أن $0 \le u_n \le 1$ $(n \in \mathbb{N})$
- لنبين أن $0 \le u_{n+1} \le 1$

* $2 + u_n > 0$ و $u_n \ge 0 \Rightarrow u_n^2 \ge 0$

$$\Rightarrow \frac{u_n^2}{2+u_n} \ge 0$$

$$\Rightarrow u_{n+1} \ge 0 \qquad (1)$$

*

$$u_{n+1} - 1 = \frac{u_n^2}{2+u_n} - 1$$

$$= \frac{u_n^2 - u_n - 2}{2+u_n}$$

$$= \frac{(u_n+1)(u_n-2)}{2+u_n}$$

$$0 \le u_n \le 1 \Rightarrow 1 + u_n > 0 \quad , \quad u_n - 2 < 0 \quad , \quad 2 + u_n > 0$$

$$\Rightarrow \frac{(u_n+1)(u_n-2)}{2+u_n} \le 0$$

$$\Rightarrow u_{n+1} \le 1 \qquad (2)$$

من (1) و (2) نستنتج أن $0 \le u_{n+1} \le 1$

إذن :

$$\boxed{\forall n \in \mathbb{N} \quad ; \quad 0 \le u_n \le 1}$$

2) *

$$u_{n+1} - u_n = \frac{u_n^2}{2+u_n} - u_n$$

$$= \frac{u_n^2 - 2u_n - u_n^2}{2+u_n}$$

$$= \frac{-2u_n}{2+u_n}$$

بما أن $u_n \geq 0$ فإن $\dfrac{-2u_n}{2+u_n} \leq 0$

أي $u_{n+1} - u_n \leq 0$

إذن $\boxed{\forall n \in \mathbb{N} \quad ; \quad u_{n+1} \leq u_n}$

ومنه فإن (u_n) متتالية تناقصية.

* بما أن (u_n) تناقصية ومصغورة بالعدد 0 فإنها متقاربة.

3) أ- $\dfrac{1}{2}u_n - u_{n+1} = \dfrac{1}{2}u_n - \dfrac{u_n^2}{2+u_n}$

$$= \frac{(2+u_n)\,u_n - 2u_n^2}{2(2+u_n)}$$

$$= \frac{2u_n + u_n^2 - 2u_n^2}{2(2+u_n)}$$

$$= \frac{(2-u_n)\,u_n}{2(2+u_n)} \geq 0$$ (لأن $0 \leq u_n \leq 1$)

إذن $\boxed{\forall n \in \mathbb{N} \quad ; \quad u_{n+1} \leq \frac{1}{2}u_n}$

ب-

$$0 \leq u_1 \leq \frac{1}{2}u_0$$

x

$$0 \leq u_2 \leq \frac{1}{2}u_1$$

x

$$0 \leq u_3 \leq \frac{1}{2}u_2$$

x ⋮

$$0 \leq u_n \leq \frac{1}{2}u_{n-1}$$

$$0 \leq u_n \leq \left(\frac{1}{2}\right)^n u_0$$

إذن : $\boxed{\forall n \in \mathbb{N} \quad ; \quad u_n \leq \left(\frac{1}{2}\right)^n}$

ج- بما أن $\lim\limits_{n \to +\infty} \left(\frac{1}{2}\right)^n = 0$ (لأن $-1 < \frac{1}{2} < 1$)

و $0 \leq u_n \leq \left(\frac{1}{2}\right)^n$

فإن $\boxed{\lim\limits_{n \to +\infty} u_n = 0}$